AN ESTIMATE OF THE INTERACTION OF A CHECKERBOARD ARRAY OF CRACKS

V. Z. Parton

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 94-97, 1965

All materials contain a large number of microdefects of various types whose development under the applied stress field results in the formation of a system of cracks. The nature of the interaction of the cracks in this system may vary greatly and its investigation is a matter of considerable interest. Sel'dovich noted that a checkerboard arrangement of cracks (elastic plane weakened by a doubly-periodic system of cracks of equal length) must in certain conditions result in mutual strengthening.

In the present paper this problem is investigated on the basis of a numerical solution of the problem of the theory of elasticity in approximate form and the correctness of the assumption is substantiated.

1. Let us consider a doubly-periodic system of cracks with crack length $2l$. Let ω_1, ω_2 be the main periods, D the region occupied by the body, and M_{kk} the contour of the crack with center at point $P = k\omega_1 + k'\omega_2$ ($k, k' = 0, \pm 1, \pm \ldots$) (Fig. 1). The net is symmetrical with respect to the x and y axes, and each crack is subjected to a uniform tensile stress p_0.

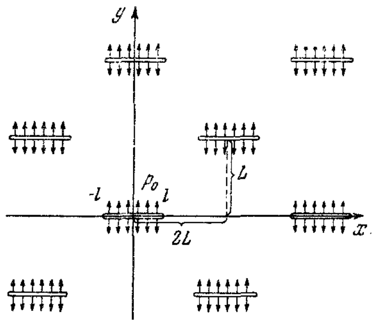

Fig. 1.

It is suggested in [1] that this problem can be reduced to finding two functions $\Phi(z)$ and $\Psi(z)$, regular with respect to D, which satisfy the following boundary condition

$$\overline{\Phi(\tau)} + \Phi(\tau) + \bar{\tau}\Phi'(\tau) + \Psi(\tau) = -p_0 \,.$$

Let us now construct an approximate solution of this problem. For this purpose we consider a single linear slot in an infinite plate, the edge of which is subjected to a uniform load p_0. In this case [1]

$$\Phi(z) = \varphi'(z) = \frac{p_0 l^2}{2\sqrt{z^2 - l^2}\,(z + \sqrt{z^2 - l^2})},$$

$$\Psi(z) = \psi'(z) = \frac{p_0 l^2 z}{2\,(z^2 - l^2)^{3/2}}, \qquad (1)$$

while the components of the stress tensor are

$$\frac{X_x}{4p_0 l^2} = \left\{\left[\chi + 4r^2\chi^{1/2} + 4r\chi^{3/4}\cos\left(\varphi - \frac{\alpha}{2}\right)\right] \times \right.$$

$$\left. \times \left[1 + \left(\frac{\chi^{1/4}\sin\alpha + 2r\sin(\varphi + 1/2\,\alpha)}{\chi^{1/4}\cos\alpha + 2r\cos(\varphi + 1/2\,\alpha)}\right)^2\right]\right\}^{-1/2} -$$

$$- 2r\chi^{-3/4}\sin\varphi\sin\frac{3\alpha}{2},$$

$$\frac{Y_y}{4p_0 l^2} = \left\{\left[\chi + 4r^2\chi^{1/2} + 4r\chi^{3/4}\cos\left(\varphi - \frac{\alpha}{2}\right)\right] \times \right.$$

$$\left. \times \left[1 + \left(\frac{\chi^{1/4}\sin\alpha + 2r\sin(\varphi + 1/2\,\alpha)}{\chi^{1/4}\cos\alpha + 2r\cos(\varphi + 1/2\,\alpha)}\right)^2\right]\right\}^{-1/2} +$$

$$+ 2r\chi^{-3/4}\sin\varphi\sin\frac{3\alpha}{2}. \qquad (2)$$

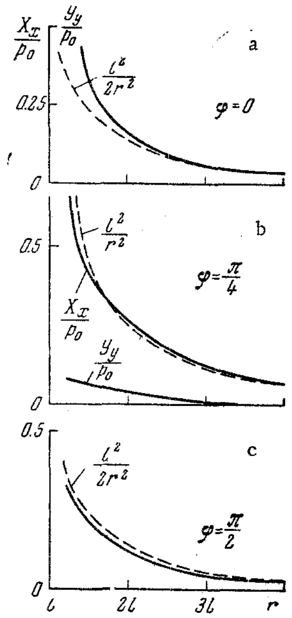

Fig. 2

Here

$$\chi = 16\,(r^4 - 2r^2 l^2 \cos 2\varphi + l^4),\ \alpha = \mathrm{arc\,tg}\,\frac{r^2\sin 2\varphi}{r^2\cos 2\varphi - l^2}. \qquad (3)$$

It is easy to see that as $|z| \to \infty$ the expressions (1) have the following asymptotic representations

$$\Phi(z) = \frac{p_0 l^2}{4z^2}, \qquad \Psi(z) = \frac{p_0 l^2}{2z^2}.$$

Figure 2 gives the relationships between the stresses calculated for unit applied load and the distance r for a given direction φ. The broken line represents the asymptotic value. It is easy to see that the true value of the stress at a given point is approximated with sufficient accuracy by the asymptotes in any direction even for points where the distance from the crack is of the order of the crack length.[1]

Thus, it is now possible to reduce the initial problem to the following form. One of the cracks, the edge of which is subjected to a uniform tensile stress p_0, with center at the origin of the coordinate system is located in an infinite body, in which operates a system of dipoles with centers at points $P = k\omega_1 + k'\omega_2$ ($k, k' = 0, \pm 1, \pm \ldots$). Since a single crack extends along the real axis, the determination of the total stress involves only the function $\Phi(z) = c/z^2$ (the value of the constant $z = 1/4\, p_0 l^2$ ensures the necessary decrease of the stresses at infinity for this problem).

For a field with a period rectangle $\beta = k\omega_1 + k'\omega_2 i$, where the stress function c/z^2 is given at each point, determination of the total stress function leads to an elliptic Weierstrass function

$$\wp(z) = \frac{c}{z^2} + c\sum_{k,k'}{}' \left[\frac{1}{(z - \beta)^2} - \frac{1}{\beta^2}\right], \qquad (4)$$

where Σ' denotes summation over all the indices at the same time with the exception of $k = k' = 0$.

[1] Asymptotic expressions for "distant" and "close" rows of parallel cracks are given by Koiter in [2].

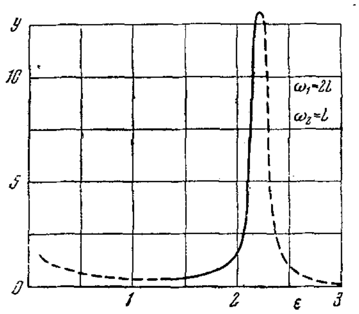

Fig. 3

Eliminating the coordinate origin, we obtain the stress function for the field outside the crack

$$F(z)=\wp(z)-\frac{c}{z^2}=c\sum_{k,\,k'}{}'\left[\frac{1}{(z-\beta)^2}-\frac{1}{\beta^2}\right]. \tag{5}$$

Along the real x axis we have

$$X_x=Y_y=2\,\mathrm{Re}\,F(z)$$

$$F(z)=\wp(z)-\frac{c}{z^2}=c\sum_{k,\,k'}{}'\left[\frac{1}{(x-\beta)^2}-\frac{1}{\beta^2}\right]=$$

$$=2c\sum_{k>0}^{\infty}\sum_{k'>0}^{\infty}\left\{\frac{(x-k\omega_1)^2-k'^2\omega_2^2}{[(x-k\omega_1)^2+k'^2\omega_2^2]^2}+\right.$$

$$\left.+\frac{(x+k\omega_1)^2-k'^2\omega_2^2}{[(x+k\omega_1)^2+k'^2\omega_2^2]^2}-\frac{2(k^2\omega_1^2-k'^2\omega_2^2)}{(k^2\omega_1^2+k'^2\omega_2^2)^2}\right\},$$

$$Y_y=4c\sum_{k>0}^{\infty}\sum_{k'>0}^{\infty}\left\{\frac{(x-k\omega_1)^2-k'^2\omega_2^2}{[(x-k\omega_1)^2+k'^2\omega_2^2]^2}+\right.$$

$$\left.+\frac{(x+k\omega_1)^2-k'^2\omega_2^2}{[(x+k\omega_1)^2+k'^2\omega_2^2]^2}-\frac{2(k^2\omega_1^2-k'^2\omega_2^2)}{(k^2\omega_1^2+k'^2\omega_2^2)^2}\right\}. \tag{6}$$

The half-length l of an isolated symmetrical crack in dynamic equilibrium is determined from the equation (see [3])

$$\int_0^l\frac{p(x)\,dx}{\sqrt{l^2-x^2}}=\frac{K}{\sqrt{2l}}\,. \tag{7}$$

Here k is the cohesion modulus [3], and p(x) is the distribution of normal stresses in the uncracked body under the same load.

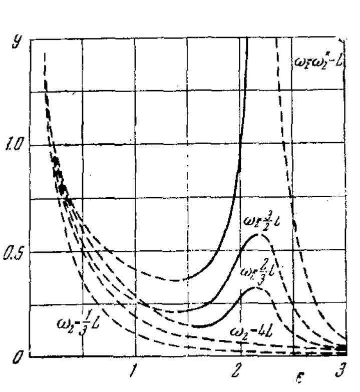

Fig. 4

In the case considered

$$p(x)=p_0+Y_y, \tag{8}$$

while Y_y is given by equation (6).

Substituting (8) into (7) and integrating, after first interchanging the summation and integration signs, we get in dimensionless parameters

$$\left\{\sum_{k=2}^{\infty}\left[\frac{2k}{(4k^2-\varepsilon^2)^{3/2}}-\frac{1}{4k^2}\right]+\sum_{k'=2}^{\infty}\left[\frac{1}{k'^2}-\frac{k'}{(\varepsilon^2+k'^2)^{3/2}}\right]+\right.$$

$$+4\varepsilon^2\sum_{k=1}^{\infty}\sum_{k'=1}^{\infty}\left[\frac{kA^2S+kAST+kk'^2(A+B)T-48k^3k'^2A}{A^3\,S\,\sqrt{A-B+8k^2}}-\right. \tag{9}$$

$$\left.\left.-\frac{4k^2-k'^2}{2\sqrt{2}(4k^2+k'^2)^2}\right]\right\}\pi\sqrt{\varepsilon}=\frac{1}{y}\qquad\begin{pmatrix}S=A+B-8k^2\\ T=16k^2+A-2B\end{pmatrix}$$

$$(B=k'^2+4k^2+\varepsilon^2,\quad A=\sqrt{B^2-16k^2\varepsilon^2},\quad \varepsilon=l/L\,,\quad y=p_0\sqrt{L}/K).$$

Summation in the ordinary sums is only over even k and k', and in the double sums over all k and k', but so that the sum of these indices is even.

Here for the sake of simplicity we have set $\omega_1=2L$, $\omega_2=L$. In order to obtain solutions for any side lengths of the period rectangle in (9), after the summation sign k must be replaced by $k\omega_1/2L$ and k' by $k'\omega_2/L$.

Relation (9) for $y=y(\varepsilon)$ is given in Fig. 3 with $\omega_1=2L$, $\omega_2=L$, while Fig. 4 gives the relations for various periods ω_2 ($\omega_1=2L$ is fixed). These curves suggest certain conclusions.

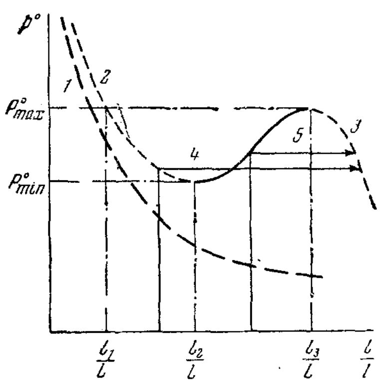

Fig. 5

The crack is stable if the stress p_0 needed to maintain it in dynamic equilibrium increases with increasing crack length $2l$. There exists a certain optimum value ω_2^* (with ω_1 constant) of the side of the period rectangle at which the stress needed to maintain the crack in dynamic equilibrium reaches a maximum. In this case the length of the stable section of the curve is much larger than in the case of a crack reinforced by stiffening ribs which prevent it from spreading [4]. A reduction or increase of ω_2 compared with ω_2^* leads to a reduction of the maximum value of the stress until ω_2 becomes equal to certain critical values ω_{21} ($\omega_{21}<\omega_2^*$) and ω_{22} ($\omega_{22}>\omega_2^*$). At $\omega_2>\omega_{22}$ and $\omega_2<\omega_{21}$ the curve has no growth segments.

Thus, one crack with initial length $2l_0$ in a field of cracks of the same length may develop, with increasing p_0, in accordance with one of the variants considered (Fig. 5).

If the curve corresponding to the given ω_2 has no stable section ($\omega_{22}<\omega_2<\omega_{21}$), then increase of the load p_0 does not affect the length $2l_0$ until the crack enters the state of dynamic equilibrium. After a corresponding load has been reached, the crack begins to propagate catastrophically, and the body is destroyed. The development of a crack is represented in this case by curve 1 in Fig. 5. If the curve has a stable section ($\omega_{21}<\omega_2<\omega_{22}$), then at $l_0<l_1$ and $l_0>l_3$ the development of the initial crack takes place in the same way as in the former case (curves 2, 3, in Fig. 5). If $l_1<l_0<l_2$, then the crack length does not change, until the crack enters the state of dynamic equilibrium. As soon as this stage is reached, the smallest increase of the equilibrium load causes the crack to go over into another stable state of dynamic equilibrium, corresponding to

the same load, whereupon it develops stably with increasing load p_0 until the load $p_{0\max}$ is reached. Beyond this value the crack begins to spread catastrophically, and failure ensues. The development of the crack in this case is represented by curve 4. At $l_2 < l_0 < l_3$ the size of the crack does not change until it enters the state of dynamic equilibrium, whereupon an increase in load causes stable development of the crack until the load reaches the value $p_{0\max}$ which causes failure (curve 5). Thus, a particular distribution of the cracks results in their stabilization.

The deformation of an infinite plate with a crack and singularities distributed in checkerboard fashion can be described by means of the following dislocation model. The crack is either a system of edge dislocations with opposite signs, in the simplest case with the Burgers vector pointing along its length, or a system of pairs of dislocations with opposite signs. The singularities can be represented as a network of vacancies or dislocation dipoles, since each singularity is considered to be neutral. The stationary vacancies affect the propagation of the dislocation-crack. The result obtained shows that for a certain density and corresponding distribution of the vacancies the material becomes stronger.

The author is indebted to G. I. Barenblatt for supervision and assistance.

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11 May 1965 Moscow